

$y = e^{-x}$ is a solution of

$$xy'' + (2x+1)y' + (x+1)y = 0.$$

SCORE: ____ / 40 PTS

Find the general solution of

$$xy'' + (2x+1)y' + (x+1)y = \frac{1}{xe^x}.$$

$$g = \frac{1}{x^2 e^x} = x^{-2} e^{-x}$$

$$y_2 = v e^{-x}$$

$$y_2' = v' e^{-x} - v e^{-x}$$

$$y_2'' = v'' e^{-x} - 2v' e^{-x} + v e^{-x}$$

$$e^{-x}(v'' x - v'(2x) + v(x) + v'(2x+1) - v(2x+1) + v(x+1)) = 0$$

$$\textcircled{4} \quad v'' x + v' = 0 \rightarrow u = v'$$

$$\frac{du}{dx} = v''$$

$$x \frac{du}{dx} + u = 0$$

$$\int \frac{1}{u} du = \int -\frac{1}{x} dx$$

$$\ln|u| = -\ln|x|$$

$$v' = u = x^{-1}$$

$$v = \ln|x|$$

$$y_2 = e^{-x} \ln|x|$$

$\textcircled{2}$ EACH
EXCEPT
AS
NOTED

$$W = \begin{vmatrix} e^{-x} & e^{-x} \ln|x| \\ -e^{-x} & -e^{-x} \ln|x| + x^{-1} e^{-x} \end{vmatrix} = x^{-1} e^{-2x}$$

$$y_p = -e^{-x} \int \frac{x^{-2} e^{-x} e^{-x} \ln|x|}{x^{-1} e^{-2x}} dx + e^{-x} \ln|x| \int \frac{x^{-2} e^{-x} e^{-x}}{x^{-1} e^{-2x}} dx \quad \textcircled{4}$$

$$= -e^{-x} \int \frac{\ln|x|}{x} dx + e^{-x} \ln|x| \int \frac{1}{x} dx$$

$$= -e^{-x} \cdot \frac{1}{2} (\ln|x|)^2 + e^{-x} (\ln|x|)^2 = \frac{1}{2} e^{-x} (\ln|x|)^2$$

$$y = \frac{1}{2} e^{-x} (\ln|x|)^2 + c_1 e^{-x} + c_2 e^{-x} \ln|x|$$

THIS IS A TYPO - WAS SUPPOSED

$y = x$ is a solution of

$$4x^2 y'' + 5xy' + y = 6x.$$

$y = x^2$ is a solution of

$$4x^2 y'' + 5xy' + y = 19x^2.$$

Solve the initial value problem

$$4x^2 y'' + 5xy' + y = 3x - 38x^2, \quad y(1) = -1, \quad y'(1) = -3$$

TO BE

8

SCORE: ____ / 25 PTS

$$L \rightarrow = \frac{1}{2}(6x) - 2(19x^2)$$

$$y_p = \frac{1}{2}x - 2x^2$$

$$4r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{-15}}{8} = -\frac{1}{8} \pm \frac{\sqrt{15}}{8}i$$

③ EACH

$$y = \frac{1}{2}x - 2x^2 + C_1 x^{-\frac{1}{8}} \cos\left(\frac{\sqrt{15}}{8} \ln x\right) + C_2 x^{-\frac{1}{8}} \sin\left(\frac{\sqrt{15}}{8} \ln x\right)$$

$$-1 = \frac{1}{2} - 2 + C_1 \rightarrow C_1 = \frac{1}{2}$$

$$y' = \frac{1}{2} - 4x - \frac{1}{8}C_1 x^{-\frac{9}{8}} \cos\left(\frac{\sqrt{15}}{8} \ln x\right) - \frac{\sqrt{15}}{8}C_1 x^{-\frac{9}{8}} \sin\left(\frac{\sqrt{15}}{8} \ln x\right) + \frac{\sqrt{15}}{8}C_2 x^{-\frac{9}{8}} \cos\left(\frac{\sqrt{15}}{8} \ln x\right) - \frac{1}{8}C_2 x^{-\frac{9}{8}} \sin\left(\frac{\sqrt{15}}{8} \ln x\right)$$

$$-3 = \frac{1}{2} - 4 - \frac{1}{16} + \frac{\sqrt{15}}{8}C_2 \rightarrow \frac{\sqrt{15}}{8}C_2 = \frac{9}{16}$$

$$C_2 = \frac{9}{2\sqrt{15}} = \frac{9\sqrt{15}}{30} = \frac{3\sqrt{15}}{10}$$

$$y = \frac{1}{2}x - 2x^2 + \frac{1}{2}x^{-\frac{1}{8}} \cos\left(\frac{\sqrt{15}}{8} \ln x\right) + \frac{3\sqrt{15}}{10}x^{-\frac{1}{8}} \sin\left(\frac{\sqrt{15}}{8} \ln x\right)$$

⑧ EXTRA POINTS DUE TO ADDED COMPLEXITY
FROM TYPO

Use elimination (as shown in lecture) to solve the system

SCORE: ____ / 45 PTS

$$\textcircled{1} (D+2)[x] + (3D+4)[y] = 2t$$

$$\textcircled{2} (2D+3)[x] + (5D+6)[y] = 3t$$

APPLY $(2D+3)$ TO $\textcircled{1}$

$-(D+2)$ TO $\textcircled{2}$

$$((2D+3)(3D+4) - (D+2)(5D+6))[y] = 4 + 6t - (3 + 6t)$$

$\textcircled{2}$ EACH
EXCEPT
AS
NOTED

$$\textcircled{4} (D^2 + D)[y] = 1 \textcircled{4}$$

$$r = 0, -1$$

$$y_h = c_1 + c_2 e^{-t}$$

$$y_p = At$$

$$y_p' = A$$

$$y_p'' = 0$$

$$y_p'' + y_p' = A = 1$$

$$y = t + c_1 + c_2 e^{-t}$$

APPLY $-(5D+6)$ TO $\textcircled{1}$

$(3D+4)$ TO $\textcircled{2}$

$$(D^2 + D)[x] = -(10 + 12t) + (9 + 12t) = -1 \textcircled{4}$$

$$x = -t + k_1 + k_2 e^{-t}$$

$$\textcircled{1} \quad -1 - k_2 e^{-t}$$

$$-2t + 2k_1 + 2k_2 e^{-t}$$

$$3 - 3c_2 e^{-t}$$

$$4t + 4c_1 + 4c_2 e^{-t} = 2t$$

$$4c_1 + 2k_1 + 2 = 0 \rightarrow k_1 = -2c_1 - 1$$

$$c_2 + k_2 = 0 \rightarrow k_2 = -c_2$$

$$x = \textcircled{1} -t - (2c_1 + 1) - c_2 e^{-t}$$

$$y = t + c_1 + c_2 e^{-t}$$

$$\rightarrow (4c_1 + 2k_1 + 2) + (c_2 + k_2)e^{-t} = 0$$

Find the general solution of $y^{(4)} - y = 24 \sin t$.

SCORE: ____ / 40 PTS

$$r^4 - 1 = 0$$

$$(r^2 - 1)(r^2 + 1) = 0$$

$$r = \pm 1, \pm i$$

(4) EACH

$$y_h = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$

$$y_p = At \cos t + Bt \sin t$$

$$y_p' = A \cos t - At \sin t + Bt \cos t + B \sin t = (A + Bt) \cos t + (B - At) \sin t$$

$$y_p'' = B \cos t + (-A - Bt) \sin t + (B - At) \cos t - A \sin t = (2B - At) \cos t + (-2A - Bt) \sin t$$

$$y_p''' = -A \cos t + (-2B + At) \sin t + (-2A - Bt) \cos t - B \sin t = (-3A - Bt) \cos t + (-3B + At) \sin t$$

$$y_p^{(4)} = -B \cos t + (3A + Bt) \sin t + (-3B + At) \cos t + A \sin t = (-4B + At) \cos t + (4A + Bt) \sin t$$

$$y_p^{(4)} - y_p = -4B \cos t + 4A \sin t$$

$$-4B = 0 \quad 4A = 24$$

$$A = 6$$

$$y = 6t \cos t + c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$